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Parametric S-matrix fluctuations in the quantum theory of chaotic scattering

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We study the effects of an arbitrary external perturbation in the statistical properties of the S matrix of quantum chaotic scattering systems in the limit of isolated resonances. We derive, using supersymmetry, an exact nonperturbative expression for the parameter dependent autocorrelator of two S-matrix elements. Universality is obtained by appropriate rescaling of the physical parameters. We propose this universal function as a signature of quantum chaos in open systems.

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Quantum chaotic scattering is the study of quantum transport in systems whose underlying classical dynamics is chaotic. It has been recently the subject of intense theoretical and experimental investigations [1]. This growing interest has been partially motivated by a large number of applications in many branches of physics, such as microwave transmission through irregular shaped cavities [2,3], ballistic transport in semiconductor nanostructures [4], and resonant reaction theory in molecular and nuclear physics [5].

A generic setup for a two-probe quantum chaotic scattering problem consists of an interaction region of finite volume (the resonant cavity in a microwave experiment or the ballistic microstructure in mesoscopic physics) connected to two reservoirs by free-propagation regions (waveguides for microwaves or perfectly conducting leads for electron waves) wherein asymptotic scattering channels can be defined. The principal role of the interaction region is to provide a mechanism for "trapping" the incoming waves by irregular boundary scattering thereby driving the system to a regime where the ray optic limit (or classical dynamics) is dominated by classical chaos.

There are two main theoretical descriptions of this problem: the semiclassical treatment [1] and the stochastic approach [6,7]. In the semiclassical description, averages are calculated by representing the S matrix as a sum over all classical trajectories connecting two given scattering channels [8]. Predictions are quantitatively accurate only for systems containing a large number of open channels. In the stochastic approach the calculation of averages over actual parameters of the physical system is replaced by averages over an ensemble of random matrices (the Hamiltonian describing the interaction region or the S matrix of the whole scattering problem). This procedure, which relies on ergodicity, can only be justified on time scales sufficiently large so that chaotic scattering dominates and the incoming waves form long-lived resonances in the interaction region.

The use of random matrix theory (RMT) in quantum chaos is now quite widespread and has been largely motivated by the fact that it provides a natural framework for obtaining quantum signatures of chaotic behavior. A striking example is the Wigner-Dyson statistics [9] for level spacing distribution, which owing to its remarkable robustness and universality can be considered the hallmark of quantum chaos [10].

In a very interesting recent series of papers [11–13] Simons, Altshuler, Lee, and Szafer have extended RMT to describe the response of the spectra of disordered and quantum chaotic systems to an external adiabatic perturbation that causes the levels to disperse in disjoint manifolds exhibiting many avoided crossings as the perturbation parameter varies. It has been demonstrated that the n-point function for density of states fluctuations becomes a universal function if the system dependent parameters are removed by appropriate rescalings. More generally, their results can be interpreted as a signature of quantum chaos in closed systems.

For open systems, such as the quantum scattering setup described above, the injection and emission of waves through the contacts provides a mechanism for level broadening and the above authors' analysis does not apply. In this case, the search for new quantum signatures of chaotic behavior is still open and is the main objective of the present work.

More precisely, we study the effects of an external adiabatic perturbation on the fluctuation pattern of elements of the random S matrix describing quantum transport through a chaotic cavity weakly coupled to external reservoirs. We demonstrate, by explicit calculation of the ensemble average using supersymmetry, that the two-point autocorrelator of two S-matrix elements at different values of the external perturbation parameter U becomes a universal expression if U is measured in units of the mean square gradient of the energy levels and if the decay width Γ is measured in units of the mean level spacing. We consider both systems with and without time reversal symmetry (T symmetry).

The technique used in the present work permits, in fact, the complete solution of the problem for an arbitrary number of channels and strength of the couplings to the external reservoirs. The motivation to restrict our analysis to the particular case of weak coupling is twofold. First, the final expression for the two-point function simplifies enormously and consequently the physical understanding of each component becomes straightforward. Second, this regime is highly nonperturbative and thus allows a deeper comprehension of the limitations of a semiclassical treatment. The complete analysis of the more general case is planned to be discussed elsewhere [14].

A general two-probe quantum chaotic scattering problem can be explicitly set up as follows. Let $|\psi_a^c(E)\rangle$ represent scattering eigenfunctions inside the right ($c=R$) and left

($c=L$) free-propagation regions, in which $a=1,2,\ldots,\Lambda$ labels the physical scattering channels. We denote the complete set of orthonormal states characterizing the interaction region by $|\mu\rangle$ and thus the Hamiltonian of the whole system, written in the basis $\{|\psi_a^c(E)\rangle,|\mu\rangle\}$, is given by [7]

$$\mathcal{H}_\beta=\sum_{ac}\int dE|\psi_a^c(E)\rangle E\langle\psi_a^c(E)|+\sum_{\mu\nu}|\mu\rangle(H_\beta(U))_{\mu\nu}\langle\nu|$$
$$+\sum_{\mu ac}\int dE\{|\mu\rangle W_{\mu a}^c\langle\psi_a^c(E)|+\text{H.c.}\},\qquad(1)$$

where $H_\beta(U)$ represents the projection of the full Hamiltonian onto the interaction region. We assume that $H_\beta(U)$ can be written as [13] $H_\beta(U)=H_\beta(0)+UV_\beta$, where $H_\beta(0)$ is a member of the Gaussian unitary ensemble (GUE) for systems without T symmetry ($\beta=2$) and belongs to the Gaussian orthogonal ensemble (GOE) for systems with T symmetry ($\beta=1$), while V_β is a fixed traceless matrix with the same symmetries as $H_\beta(0)$. Two important points should be made here. Firstly, we are making the conjecture (following Lewenkopf and Weidenmüller [6]) that if our quantum mechanical scattering problem has a classical counterpart which is fully chaotic and with all parts of the phase space equally accessible, then the matrix $H_\beta(0)$ belongs to one of the Gaussian ensembles of random matrices. Secondly, it is natural to argue that the appropriate statistical assumption for V_β is that it should also be a random matrix belonging to the same ensemble as $H_\beta(0)$. However, so long as we are only interested in universal functions (functions with rescaled arguments), it can be shown [13,15] that the choice of V_β as a fixed traceless matrix leads to the same universal (after appropriate rescaling) nonlinear σ model. The third term in (1) represents the coupling between the eigenstates of the interaction region and the scattering states in the free-propagation region.

For any finite number N of bound states in the interaction region (we shall later take $N\rightarrow\infty$ at the end of the calculation) the kernel of the Lippmann-Schwinger equation is of finite rank, thus the S matrix of the problem can be calculated algebraically [16] and after some straightforward simplifications yields

$$S_{ab}^{cc'}(U)=\delta^{cc'}\delta_{ab}-2i\pi\sum_{\mu\nu}W_{\mu a}^c(D^{-1})_{\mu\nu}W_{\nu b}^{c'},\qquad(2)$$

in which

$$D_{\mu\nu}\equiv(E+i0^+)\delta_{\mu\nu}-[H_\beta(U)]_{\mu\nu}+i\pi\sum_{ac}W_{\mu a}^cW_{\nu a}^c.$$

The absence of direct transitions between the physical channels enables us to work in a representation that satisfies the requirement $\Sigma_\mu W_{\mu a}^cW_{\mu b}^{c'}=N\delta^{cc'}\delta_{ab}x_a^c$. The weak coupling regime is obtained simply by requiring $x_a^c\ll\Delta$, where Δ is the mean level spacing. Using Eq. (2) one can show that the transmission probabilities defined as $T_a^c\equiv1-|\langle S_{aa}^{cc}\rangle|^2$ are given by $T_a^c\sim4\pi^2x_a^c/\Delta\ll1$, which physically corresponds to the limit of weakly overlapping levels, where transport is dominated by isolated resonances. The relevance of this regime in the context of mesoscopic conductors has recently been discussed by Prigodin, Efetov, and Iida [17] and by Zirnbauer [18].

The simplest two-point function that contains sufficient information about the perturbation driven fluctuations in the elements of the S matrix of the problem is given by

$$\chi_\beta(U)=\sum_{abcc'}\langle S_{ab}^{cc'}(\bar{u})S_{ab}^{cc'*}(\bar{u}+U)\rangle,\qquad(3)$$

where the angular brackets denote the usual ensemble average. We remark that this correlation function has recently been discussed by Doron, Smilansky, and Frenkel [3] using a semiclassical approach. Their result, however, is not valid in the weak coupling regime considered in the present work, where pure quantum effects are dominant and classical dynamics is irrelevant.

As an example of a direct observation of a correlation function similar to (3) but taken as a function of energy in the absence of an external perturbation we mention the microwave experiment by Doron, Smilansky, and Frenkel [2]. Finally it is interesting to observe that for applications in mesoscopic physics the average Landauer conductance is simply given by $\bar{G}_\beta=(e^2/h)\chi_\beta(0)$.

We now state our results. Taking N and Λ to infinity such that $T_a^c\rightarrow0$, but with $T^c=\Sigma_aT_a^c$ remaining finite [19], $\chi_\beta(U)$ can then be calculated exactly using the conventional mapping [7,20] of RMT onto the zero-dimensional nonlinear supersymmetric σ model. Integrating over the compact manifold of the massless transverse modes by means of Efetov's parametrization [20] of the auxiliary supermatrix Q fields, we find

$$\chi_\beta(U)=\frac{\pi}{\Delta}\Gamma TI_\beta\left(\frac{\pi}{\Delta}\Gamma,\frac{U\pi}{2}(\beta C_0)^{1/2}\right),\qquad(4)$$

where $T\equiv(1/T^R+1/T^L)^{-1}$ is the total transmission coefficient across the interaction region, $\Gamma\equiv\Delta(T^R+T^L)/(2\pi)$ is the total decay width for wave emission into the free-propagation region, and C_0 is the average gradient of level velocities as defined in Refs. [11–13]. We remark that Eq. (4) can also be derived from a microscopic model of a particle diffusing in a disordered potential by confining the saddle point Lagrangian to the lowest harmonic in the spatial dependence of the composite supermatrix fields using the technique of Refs. [20,11]. Finally, the function $I_\beta(x,y)$ can be written as

$$I_\beta(x,y)=\int_{(\beta)}d\underline{\lambda}\,\mu_\beta(\underline{\lambda})f_\beta(\underline{\lambda})\exp[-xg_\beta(\underline{\lambda})-y^2f_\beta(\underline{\lambda})],$$

in which

$$\int_{(1)}d\underline{\lambda}=\int_1^\infty d\lambda_1\int_1^\infty d\lambda_2\int_{-1}^1 d\lambda_3,$$

$$\int_{(2)}d\underline{\lambda}=\int_1^\infty d\lambda_1\int_{-1}^1 d\lambda_2,$$

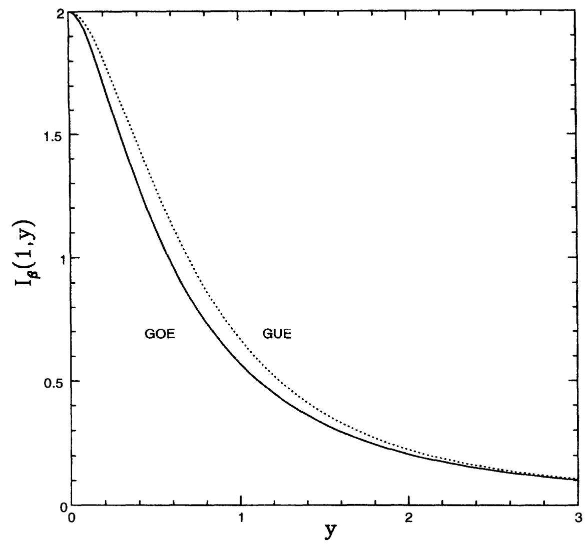

FIG. 1. $I_\beta(x,y)$ for $x=1$ as a function of y for $\beta=1$ (GOE) and $\beta=2$ (GUE) as indicated.

$$\mu_1(\underline{\lambda})=\frac{(1-\lambda_3^2)}{(\lambda_1^2+\lambda_2^2+\lambda_3^2-2\lambda_1\lambda_2\lambda_3-1)^2},$$

$$f_1(\underline{\lambda})=2\lambda_1^2\lambda_2^2-\lambda_1^2-\lambda_2^2-\lambda_3^2+1,$$

$$\mu_2(\underline{\lambda})=(\lambda_1-\lambda_2)^{-2},\quad f_2(\underline{\lambda})=\lambda_1^2-\lambda_2^2,$$

$$g_1(\underline{\lambda})=\lambda_1\lambda_2-\lambda_3,\quad g_2(\underline{\lambda})=\lambda_1-\lambda_2.$$

Observe that $\chi_\beta(U)$ becomes a universal function after the rescalings [11–13] $\Gamma\rightarrow\Delta\hat{\Gamma}$ and $U\rightarrow U_c\hat{U}$, where $U_c=C_0^{-1/2}$. Equation (4) is the complete solution of the problem and can be interpreted as a signature of quantum chaos in open systems in the weak coupling regime.

The function $I_\beta(x,y)$ is displayed in Fig. 1 for $\beta=1$ and 2. Its typical Lorentzian-like tails reflect the long range nature of the universal logarithmic eigenvalue repulsion of RMT. One can see two interesting limits: (i) $U\gg U_c$ and (ii) $U\ll U_c$.

(i) In this case the levels decorrelate asymptotically and diagrammatic perturbation theory applies. Alternatively, we can calculate $I_\beta(x,y)$ asymptotically for $y\gg1$ to find $I_\beta(x,y)\sim 1/y^2$. Thus $\chi_\beta(U)$ acquires, after rescaling, the simple universal form

$$\hat{\chi}_\beta(U)\simeq\frac{4\hat{\Gamma}T}{\pi\hat{U}^2\beta}\quad\text{for}\quad \hat{U}\gg1. \tag{5}$$

Diagrammatically, the main contributions to $\hat{\chi}_\beta(U)$ for systems with orthogonal symmetry ($\beta=1$) come from cooperon and diffuson modes of diffusion, while for systems with unitary symmetry ($\beta=2$) the cooperon degrees of freedom are destroyed by the breaking of T symmetry. Cooperon and diffuson are defined in the context of diagrammatic calculations as a sum of classes of polarization diagrams which contributes to the average of the product of two Green functions. Therefore, when we cross over from the orthogonal to the unitary ensemble there is a suppression of correlations in $\hat{\chi}_\beta(U)$ by a universal factor 2.

(ii) This case is very interesting since it corresponds to a regime where perturbation theory and semiclassical approach break down. Using our exact result, Eq. (4), for $\chi_\beta(U)$ we find

$$\chi_\beta(U)=2T\left[1-\frac{\beta\pi^2}{2\gamma}\left(\frac{U}{U_c}\right)^2 h_\beta(\gamma)+O((U/U_c)^4)\right], \tag{6}$$

in which $\gamma=\pi\Gamma/\Delta$ and

$$h_1(\gamma)=h_2(\gamma)+\frac{1}{\gamma}+\int_\gamma^\infty\frac{e^{-t}}{t}dt\left(\frac{d}{d\gamma}\frac{\sinh\gamma}{\gamma}\right),$$

where

$$h_2(\gamma)=1+\frac{1-e^{-2\gamma}}{2\gamma^2}.$$

The physical meaning of the first term in Eq. (6) becomes more transparent if we consider its application to ballistic mesoscopic nanostructures and use the definition of T and the Landauer formula for the average conductance to write

$$\bar{G}_\beta=\frac{e^2}{h}\chi_\beta(0)=2\frac{e^2}{h}T=2\frac{e^2}{h}\frac{T^RT^L}{T^R+T^L}. \tag{7}$$

This expression coincides with the average conductance of a quantum dot weakly coupled to external leads obtained in Ref. [17] using a different method. It demonstrates that transport in this regime is completely dominated by tunneling at the junctions and therefore the average Landauer conductance is just the series addition of the contact conductance associated with the couplings between the quantum dot and the bulk leads. Note that $\bar{G}_\beta$ is independent of the size L of the dot in sharp contrast with Ohm's law, where the diffusive process inside the sample dominates and $\bar{G}_\beta$ decays linearly with L. It is interesting to observe how $\bar{G}_\beta$ changes when the coupling to the leads is strengthened driving the system to the strongly absorbing regime. In this case the decay width Γ_a at each channel becomes comparable to the mean level spacing and thus $T^R=T^L\simeq\Lambda$ (in the symmetric case). Note that this regime corresponds to the limit of a large number of open channels and thus it can be treated by both conventional perturbation theory and semiclassical analysis. For systems with unitary symmetry the total transmission coefficient becomes equal to the total reflection coefficient $T=R=\Lambda/2$, as a result of classical ergodic exploration of the boundaries of the cavity and Eq. (7) becomes $\bar{G}_2=(e^2/h)\Lambda$. As discussed in Ref. [21] the presence of quantum interference in systems with orthogonal symmetry leads to a small correction due to weak localization and we find $\bar{G}_1=\bar{G}_2+\delta G$.

Finally, the second term in Eq. (6) determines the curvature $K_\beta(\gamma)$ of $\chi_\beta(U)$ at $U=0$. The functions $h_1(\gamma)$ and $h_2(\gamma)$ arise ultimately from level repulsion and eigenvector rotations induced by the external perturbation. One can verify by direct calculation that $K_1(\gamma)>K_2(\gamma)$ for all $\gamma>0$.

In the theory of Simons, Altshuler, Lee, and Szafer [11–13] of closed chaotic systems in the presence of an external perturbation a remarkable web of relations has been found [13] between the nonlinear σ model of diffusion and many other problems in theoretical physics, such as continuous matrix models, Dyson's Brownian motion model, Sutherland quantum Hamiltonian, and Pechukas gas. In the light of our result, we believe that similar relations exist for open systems. An interesting line of research, which we leave for the future, would be to build a Brownian motion model (or equivalently a quantum Hamiltonian) whose solution would describe the statistics of the elements of the S matrix in the presence of an external perturbation.

In conclusion, we have studied the effects of an adiabatic external perturbation on the correlations between different elements of the S matrix describing scattering in an open quantum chaotic system weakly coupled to external reservoirs by two free-propagating pipes. We demonstrate that the parameter dependent autocorrelator of two S-matrix elements becomes a universal expression if the perturbation parameter U is measured in units of the mean square gradient of the energy levels and if the decay width Γ is measured in units of the mean level spacing. We have proposed this universal function as a signature of quantum chaos in open systems in the weak coupling regime. We believe that our predictions should in principle be observable in a microwave-scattering experiment, where both amplitude and phase of the scattered wave can be accurately measured. The variable U could, for instance, parametrize changes in the geometry of the chaotic cavity.

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[1] For a review see U. Smilansky, in *Proceedings of the 1989 Les Houches Summer School on Chaos and Quantum Physics*, edited by M. J. Giannoni, A. Voros, and J. Zinn-Justin (North-Holland, Amsterdam, 1991), p. 371; B. Eckardt, Physica D **33**, 89 (1988).

[2] H.-J. Stöckman and J. Stein, Phys. Rev. Lett. **64**, 2215 (1990); E. Doron, U. Smilansky, and A. Frenkel, *ibid.* **65**, 3072 (1990).

[3] E. Doron, U. Smilansky, and A. Frenkel, Physica D **50**, 367 (1991).

[4] For a review see H. U. Baranger, R. A. Jalabert, and A. D. Stone, in *Transport Phenomena in Mesoscopic Systems*, edited by H. Fukuyama and T. Ando (Springer-Verlag, Berlin, 1992); C. W. J. Beenakker and H. van Houten, in *Solid State Physics*, edited by H. Ehrenreich and D. Turnbull (Academic Press, New York, 1991), Vol. 44; *Mesoscopic Phenomena in Solids*, edited by B. L. Altshuler, P. A. Lee, and R. A. Webb (North-Holland, New York, 1991).

[5] D. W. Noid, S. K. Gray, and S. A. Rice, J. Chem. Phys. **84**, 2649 (1986); W. Dünneweber *et al.*, Phys. Rev. Lett. **61**, 927 (1988); R. Bonetti and M. S. Hussein, *ibid.* **57**, 195 (1986).

[6] P. A. Mello, P. Pereyra, and T. H. Seligmann, Ann. Phys. (N.Y.) **161**, 254 (1985); C. H. Lewenkopf and H. A. Weidenmüller, *ibid.* **212**, 53 (1991).

[7] J. J. M. Verbaarschot, H. A. Weidenmüller, and M. R. Zirnbauer, Phys. Rep. **129**, 367 (1985).

[8] See, for instance, W. H. Miller, in *Advances in Chemical Physics*, edited by K. P. Lawley (Wiley, New York, 1975), Vol. 30; M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1991).

[9] E. P. Wigner, Ann. Math. **53**, 36 (1951); F. J. Dyson, J. Math. Phys. **3**, 140 (1962).

[10] See, e.g., F. Haake, *Quantum Signatures of Chaos* (Springer, Berlin, 1992).

[11] A. Szafer and B. L. Altshuler, Phys. Rev. Lett. **70**, 587 (1993); B. D. Simons and B. L. Altshuler, *ibid.* **70**, 4063 (1993); Phys. Rev. B **48**, 5422 (1993).

[12] B. D. Simons, P. A. Lee, and B. L. Altshuler, Phys. Rev. Lett. **70**, 4122 (1993).

[13] B. D. Simons, P. A. Lee, and B. L. Altshuler, Nucl. Phys. **B409**, 487 (1993); Phys. Rev. Lett. **72**, 64 (1994).

[14] A. M. S. Macêdo and J. T. Chalker (unpublished).

[15] A. Altland, S. Iida, and K. B. Efetov, J. Phys. A **26**, 3545 (1993).

[16] C. Mahaux and H. A. Weidenmüller, *Shell-Model Approach to Nuclear Reactions* (North-Holland, Amsterdam, 1969).

[17] V. N. Prigodin, K. B. Efetov, and S. Iida, Phys. Rev. Lett. **71**, 1230 (1993).

[18] M. R. Zirnbauer, Nucl. Phys. **A560**, 95 (1993).

[19] This limit has also been used by J. J. M. Verbaarshot, Ann. Phys. (N.Y.) **168**, 368 (1986) in the context of compound nucleus reactions.

[20] K. B. Efetov, Adv. Phys. **32**, 53 (1983).

[21] H. U. Baranger and P. A. Mello (unpublished); R. A. Jalabert, J.-L. Pichard, and C. W. J. Beenakker (unpublished).